

27. Miscellaneous Functions

IRENE A. STEGUN¹

Contents

	Page
27.1. Debye Functions $\int_0^x \frac{t^n dt}{e^t - 1}$	998
$n=1(1)4, x=0(.1)1.4(.2)5(.5)10, \quad 6D$	
27.2. Planck's Radiation Function $x^{-5}(e^{1/x} - 1)^{-1}$	999
$x=.05(.005).1(.01).2(.02).4(.05).9(.1)1.5(.5)3.5, \quad 3D$	
$x_{\max}, f(x_{\max}), \quad 9-10S$	
27.3. Einstein Functions	999
$\frac{x^2 e^x}{(e^x - 1)^2}, \quad \frac{x}{e^x - 1}, \quad \ln(1 - e^{-x}), \quad \frac{x}{e^x - 1} - \ln(1 - e^{-x})$	
$x=0(.05)1.5(.1)3(.2)6, \quad 5D$	
27.4. Sievert Integral $\int_0^\theta e^{-x \sec \phi} d\phi$	1000
$x=0(.1)1(.2)3(.5)10, \theta=10^\circ(10^\circ)60^\circ(15^\circ)90^\circ, \quad 6D$	
27.5. $f_m(x) = \int_0^\infty t^m e^{-t^2 - \frac{x}{t}} dt$ and Related Integrals	1001
$f_m(x), \quad m=1, 2, 3; x=0(.01).05, .1(.1)1, \quad 4D$	
$f_3(ix), \quad x=0(.2)8(.5)15(1)20, \quad 4-5D$	
27.6. $f(x) = \int_0^\infty \frac{e^{-t^2}}{t+x} dt$	1003
$f(x) + \ln x, \quad x=0(.05)1$	
$f(x), \quad x=1(.1)3(.5)8, \quad 4D$	
27.7. Dilogarithm (Spence's Integral) $f(x) = -\int_1^x \frac{\ln t}{t-1} dt$	1004
$x=0(.01).5, \quad 9D$	
27.8. Clausen's Integral and Related Summations	1005
$f(\theta) = -\int_0^\theta \ln \left(2 \sin \frac{t}{2} \right) dt = \sum_{n=1}^\infty \frac{\sin n\theta}{n^3}$	
$f(\theta) + \theta \ln \theta, \quad \theta=0^\circ(1^\circ)15^\circ$	
$f(\theta), \quad \theta=15^\circ(1^\circ)30^\circ(2^\circ)90^\circ(5^\circ)180^\circ, \quad 6D$	
27.9. Vector-Addition Coefficients $(j_1 j_2 m_1 m_2 j_1 j_2 j m)$	1006
Algebraic Expressions for $j_2=1/2, 1, 3/2, 2$	
Decimal Values for $j_2=1/2, 1, 3/2, \quad 5D$	

¹ National Bureau of Standards.

27. Miscellaneous Functions

27.1. Debye Functions

Series Representations

27.1.1

$$\int_0^x \frac{t^n dt}{e^t - 1} = x^n \left[\frac{1}{n} - \frac{x}{2(n+1)} + \sum_{k=1}^{\infty} \frac{B_{2k} x^{2k}}{(2k+n)(2k)!} \right] \quad (|x| < 2\pi, n \geq 1)$$

(For Bernoulli numbers B_{2k} , see chapter 23.)

27.1.2

$$\int_x^{\infty} \frac{t^n dt}{e^t - 1} = \sum_{k=1}^{\infty} e^{-kt} \left[\frac{x^n}{k} + \frac{nx^{n-1}}{k^2} + \frac{(n)(n-1)x^{n-2}}{k^3} + \dots + \frac{n!}{k^{n+1}} \right] \quad (x > 0, n \geq 1)$$

Relation to Riemann Zeta Function (see chapter 23)

27.1.3

$$\int_0^{\infty} \frac{t^n dt}{e^t - 1} = n! \zeta(n+1).$$

[27.1] J. A. Beattie, Six-place tables of the Debye energy and specific heat functions, *J. Math. Phys.* **6**, 1-32 (1926).

$$\frac{3}{x^3} \int_0^x \frac{y^3 dy}{e^y - 1}, \frac{12}{x^5} \left[\int_0^x \frac{y^5 dy}{e^y - 1} - \frac{3x}{e^x - 1} \right], x = 0(.01)24, \quad 6S.$$

[27.2] E. Grüneisen, Die Abhängigkeit des elektrischen Widerstandes reiner Metalle von der Temperatur, *Ann. Physik.* (5) **16**, 530-540 (1933).

$$\frac{20}{x^4} \int_0^x \frac{t^4 dt}{e^t - 1} - \frac{4x}{e^x - 1}, \quad x = 0(.1)13(.2)18(.1)20(.2)52(.4)80, \quad 4S.$$

Table 27.1

Debye Functions

x	$\frac{1}{x} \int_0^x \frac{t dt}{e^t - 1}$	$\frac{2}{x^2} \int_0^x \frac{t^2 dt}{e^t - 1}$	$\frac{3}{x^3} \int_0^x \frac{t^3 dt}{e^t - 1}$	$\frac{4}{x^4} \int_0^x \frac{t^4 dt}{e^t - 1}$
0.0	1.000000	1.000000	1.000000	1.000000
0.1	0.975278	0.967083	0.963000	0.960555
0.2	0.951111	0.934999	0.926999	0.922221
0.3	0.927498	0.903746	0.891995	0.884994
0.4	0.904437	0.873322	0.857985	0.848871
0.5	0.881927	0.843721	0.824963	0.813846
0.6	0.859964	0.814940	0.792924	0.779911
0.7	0.838545	0.786973	0.761859	0.747057
0.8	0.817665	0.759813	0.731759	0.715275
0.9	0.797320	0.733451	0.702615	0.684551
1.0	0.777505	0.707878	0.674416	0.654874
1.1	0.758213	0.683086	0.647148	0.626228
1.2	0.739438	0.659064	0.620798	0.598598
1.3	0.721173	0.635800	0.595351	0.571967
1.4	0.703412	0.613281	0.570793	0.546317
1.6	0.669366	0.570431	0.524275	0.497882
1.8	0.637235	0.530404	0.481103	0.453131
2.0	0.606947	0.493083	0.441129	0.411893
2.2	0.578427	0.458343	0.404194	0.373984
2.4	0.551596	0.426057	0.370137	0.339218
2.6	0.526375	0.396095	0.338793	0.307405
2.8	0.502682	0.368324	0.309995	0.278355
3.0	0.480435	0.342614	0.283580	0.251879
3.2	0.459555	0.318834	0.259385	0.227792
3.4	0.439962	0.296859	0.237252	0.205915
3.6	0.421580	0.276565	0.217030	0.186075
3.8	0.404332	0.257835	0.198571	0.168107
4.0	0.388148	0.240554	0.181737	0.151855
4.2	0.372958	0.224615	0.166396	0.137169
4.4	0.358696	0.209916	0.152424	0.123913
4.6	0.345301	0.196361	0.139704	0.111957
4.8	0.332713	0.183860	0.128129	0.101180
5.0	0.320876	0.172329	0.117597	0.091471
5.5	0.294240	0.147243	0.095241	0.071228
6.0	0.271260	0.126669	0.077581	0.055677
6.5	0.251331	0.109727	0.063604	0.043730
7.0	0.233948	0.095707	0.052506	0.034541
7.5	0.218698	0.084039	0.043655	0.027453
8.0	0.205239	0.074269	0.036560	0.021968
8.5	0.193294	0.066036	0.030840	0.017702
9.0	0.182633	0.059053	0.026200	0.014368
9.5	0.173068	0.053092	0.022411	0.011747
10.0	0.164443	0.047971	0.019296	0.009674

$$\left[\begin{smallmatrix} (-4)5 \\ 5 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)6 \\ 5 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)6 \\ 5 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)6 \\ 5 \end{smallmatrix} \right]$$

Planck's Radiation Function

Table 27.2

$$f(x) = x^{-5}(e^{1/x} - 1)^{-1}$$

x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0.050	0.007	0.10	4.540	0.20	21.199	0.40	8.733	0.9	0.831
0.055	0.025	0.11	6.998	0.22	20.819	0.45	6.586	1.0	0.582
0.060	0.074	0.12	9.662	0.24	19.777	0.50	5.009	1.1	0.419
0.065	0.179	0.13	12.296	0.26	18.372	0.55	3.850	1.2	0.309
0.070	0.372	0.14	14.710	0.28	16.809	0.60	2.995	1.3	0.233
0.075	0.682	0.15	16.780	0.30	15.224	0.65	2.356	1.4	0.178
0.080	1.137	0.16	18.446	0.32	13.696	0.70	1.875	1.5	0.139
0.085	1.752	0.17	19.692	0.34	12.270	0.75	1.508	2.0	0.048
0.090	2.531	0.18	20.539	0.36	10.965	0.80	1.225	2.5	0.021
0.095	3.466	0.19	21.025	0.38	9.787	0.85	1.005	3.0	0.010
0.100	4.540	0.20	21.199	0.40	8.733	0.90	0.831	3.5	0.006

$$x_{\max} = .20140\ 52353 \quad f(x_{\max}) = 21.20143\ 58.$$

$$\left[\begin{smallmatrix} (-2)2 \\ 4 \end{smallmatrix} \right] \quad \left[\begin{smallmatrix} (-2)5 \\ 5 \end{smallmatrix} \right] \quad \left[\begin{smallmatrix} (-2)8 \\ 5 \end{smallmatrix} \right] \quad \left[\begin{smallmatrix} (-2)7 \\ 5 \end{smallmatrix} \right] \quad \left[\begin{smallmatrix} (-2)1 \\ 4 \end{smallmatrix} \right]$$

[27.3] Miscellaneous Physical Tables, Planck's radiation functions and electronic functions, MT 17 (U.S. Government Printing Office, Washington, D.C., 1941).

Table I: $\frac{R_\lambda}{R_{\lambda \max}}, \frac{R_{0-\lambda}}{R_{0-\infty}}, \frac{N_\lambda}{N_{\lambda \max}}, \frac{N_{0-\lambda}}{N_{0-\infty}}$ for $\lambda T = [.05(.001).1(.005).4(.01).6(.02)1(.05)2] \text{ cm K}^\circ$.

$$R_\lambda = c_1 \lambda^{-5} (e^{c_2/\lambda T} - 1)^{-1}, \quad R_{0-\lambda} = \int_0^\lambda R_\lambda d\lambda,$$

Table II: $R_\lambda, R_{0-\lambda}, N_\lambda, N_{0-\lambda}$ ($T = 1000^\circ \text{ K}$) for $\lambda = [.5(.01)1(.05)4(.1)6(.2)10(.5)20] \text{ microns}$.

$$N_\lambda = 2\pi c \lambda^{-4} (e^{c_2/\lambda T} - 1)^{-1}, \quad N_{0-\lambda} = \int_0^\lambda N_\lambda d\lambda$$

Table III: N_λ for $\lambda = [.25(.05)1.6(.2)3(1)10] \text{ microns}$, $T = [1000^\circ (500^\circ) 3500^\circ \text{ K and } 6000^\circ \text{ K}]$.

Einstein Functions

Table 27.3

x	$\frac{x^2 e^x}{(e^x - 1)^2}$	$\frac{x}{e^x - 1}$	$\ln(1 - e^{-x})$	$\frac{x}{e^x - 1} - \ln(1 - e^{-x})$
0.00	1.00000	1.00000	$-\infty$	∞
0.05	0.99979	0.97521	-3.02063	3.99584
0.10	0.99917	0.95083	-2.35217	3.30300
0.15	0.99813	0.92687	-1.97118	2.89806
0.20	0.99667	0.90333	-1.70777	2.61110
0.25	0.99481	0.88020	-1.50869	2.38888
0.30	0.99253	0.85749	-1.35023	2.20771
0.35	0.98985	0.83519	-1.21972	2.05491
0.40	0.98677	0.81330	-1.10963	1.92293
0.45	0.98329	0.79182	-1.01508	1.80690
0.50	0.97942	0.77075	-0.93275	1.70350
0.55	0.97517	0.75008	-0.86026	1.61035
0.60	0.97053	0.72982	-0.79587	1.52569
0.65	0.96552	0.70996	-0.73824	1.44820
0.70	0.96015	0.69050	-0.68634	1.37684
0.75	0.95441	0.67144	-0.63935	1.31079
0.80	0.94833	0.65277	-0.59662	1.24939
0.85	0.94191	0.63450	-0.55759	1.19209
0.90	0.93515	0.61661	-0.52184	1.13844
0.95	0.92807	0.59910	-0.48897	1.08809
1.00	0.92067	0.58198	-0.45868	1.04065
1.05	0.91298	0.56523	-0.43069	0.99592
1.10	0.90499	0.54886	-0.40477	0.95363
1.15	0.89671	0.53285	-0.38073	0.91358
1.20	0.88817	0.51722	-0.35838	0.87560
1.25	0.87937	0.50194	-0.33758	0.83952
1.30	0.87031	0.48702	-0.31818	0.80520
1.35	0.86102	0.47245	-0.30008	0.77253
1.40	0.85151	0.45824	-0.28315	0.74139
1.45	0.84178	0.44436	-0.26732	0.71168
1.50	0.83185	0.43083	-0.25248	0.68331

$$\left[\begin{smallmatrix} (-5)5 \\ 3 \end{smallmatrix} \right] \quad \left[\begin{smallmatrix} (-5)5 \\ 3 \end{smallmatrix} \right]$$

Table 27.3

Einstein Functions

x	$\frac{x^2 e^x}{(e^x - 1)^2}$	$\frac{x}{e^x - 1}$	$\ln(1 - e^{-x})$	$\frac{x}{e^x - 1} - \ln(1 - e^{-x})$
1.6	0.81143	0.40475	-0.22552	0.63027
1.7	0.79035	0.37998	-0.20173	0.58171
1.8	0.76869	0.35646	-0.18068	0.53714
1.9	0.74657	0.33416	-0.16201	0.49617
2.0	0.72406	0.31304	-0.14541	0.45845
2.1	0.70127	0.29304	-0.13063	0.42367
2.2	0.67827	0.27414	-0.11744	0.39158
2.3	0.65515	0.25629	-0.10565	0.36194
2.4	0.63200	0.23945	-0.09510	0.33455
2.5	0.60889	0.22356	-0.08565	0.30921
2.6	0.58589	0.20861	-0.07718	0.28578
2.7	0.56307	0.19453	-0.06957	0.26410
2.8	0.54049	0.18129	-0.06274	0.24403
2.9	0.51820	0.16886	-0.05659	0.22545
3.0	0.49627	0.15719	-0.05107	0.20826
3.2	0.45363	0.13598	-0.04162	0.17760
3.4	0.41289	0.11739	-0.03394	0.15133
3.6	0.37429	0.10113	-0.02770	0.12883
3.8	0.33799	0.08695	-0.02262	0.10958
4.0	0.30409	0.07463	-0.01849	0.09311
4.2	0.27264	0.06394	-0.01511	0.07905
4.4	0.24363	0.05469	-0.01235	0.06705
4.6	0.21704	0.04671	-0.01010	0.05681
4.8	0.19277	0.03983	-0.00826	0.04809
5.0	0.17074	0.03392	-0.00676	0.04068
5.2	0.15083	0.02885	-0.00553	0.03438
5.4	0.13290	0.02450	-0.00453	0.02903
5.6	0.11683	0.02078	-0.00370	0.02449
5.8	0.10247	0.01761	-0.00303	0.02065
6.0	0.08968	0.01491	-0.00248	0.01739
	$\left[\begin{smallmatrix} (-4)3 \\ 4 \end{smallmatrix} \right]$	$\left[\begin{smallmatrix} (-4)3 \\ 4 \end{smallmatrix} \right]$	$\left[\begin{smallmatrix} (-4)4 \\ 4 \end{smallmatrix} \right]$	$\left[\begin{smallmatrix} (-4)6 \\ 4 \end{smallmatrix} \right]$

[27.4] H. L. Johnston, L. Savedoff and J. Belzer, Contributions to the thermodynamic functions by a Planck-Einstein oscillator in one degree of freedom, NAVEXOS p. 646, Office of Naval Research, Department of the Navy, Washington, D.C. (1949). Values of $x^2 e^x / (e^x - 1)^2$, $x / (e^x - 1)^{-1}$, $-\ln(1 - e^{-x})$ and $x / (e^x - 1)^{-1} - \ln(1 - e^{-x})$ for $x = 0(.001)3(.01)14.99$, 5D with first differences.

27.4. Sievert Integral

$$\int_0^\theta e^{-x \sec \phi} d\phi$$

Relation to the Error Function

27.4.1

$$\int_0^\theta e^{-x \sec \phi} d\phi \sim \sqrt{\frac{\pi}{2x}} e^{-x} \operatorname{erf}\left(\sqrt{\frac{x}{2}} \theta\right) \quad (x \rightarrow \infty)$$

(For erf, see chapter 7.)

Representation in Terms of Exponential Integrals

27.4.2

$$\int_0^\theta e^{-x \sec \phi} d\phi = \int_0^{\frac{\pi}{2}} e^{-x \sec \theta} d\theta - \sum_{k=0}^{\infty} \alpha_k (\cos \theta)^{2k+1} E_{2k+2} \left(\frac{x}{\cos \theta} \right) \quad \left(x \geq 0, 0 < \theta < \frac{\pi}{2} \right)$$

$$\alpha_0 = 1, \alpha_k = \frac{1 \cdot 3 \cdot 5 \cdots (2k-1)}{2 \cdot 4 \cdot 6 \cdots (2k)}$$

(For $E_{2k+2}(x)$, see chapter 5.)

Relation to the Integral of the Bessel Function $K_0(x)$

27.4.3

$$\int_0^{\frac{\pi}{2}} e^{-x \sec \phi} d\phi = \operatorname{Ki}_1(x) = \int_x^\infty K_0(t) dt \quad \text{where}$$

$$x^{\frac{1}{2}} e^x \operatorname{Ki}_1(x) \sim \left(\frac{1}{2} \pi \right)^{\frac{1}{2}} \left\{ 1 - \frac{5}{8x} + \frac{129}{128x^2} - \frac{2655}{1024x^3} + \frac{301035}{32768x^4} - \cdots \right\}$$

(For $\operatorname{Ki}_1(x)$, see chapter 11.)

[27.5] National Bureau of Standards, Table of the Sievert integral, Applied Math. Series— (U.S. Government Printing Office, Washington, D.C. In press).

$$x=0(.01)2(.02)5(.05)10, \theta=0^\circ(1^\circ)90^\circ, \quad 9D.$$

[27.6] R. M. Sievert, Die v-Strahlungsintensität an der Oberfläche und in der nächsten Umgebung von Radiumnadeln, Acta Radiologica 11, 239-301 (1930).

$$\int_0^\pi e^{-A \sin \phi} d\phi, \phi=30^\circ(1^\circ)90^\circ, A=0(.01).5, \quad 3D.$$

$$\text{Sievert Integral } \int_0^\pi e^{-x \sin \phi} d\phi$$

Table 27.4

$x \backslash \theta$	10°	20°	30°	40°	50°	60°	75°	90°
0.0	0.174533	0.349066	0.523599	0.698132	0.872665	1.047198	1.308997	1.570796
0.1	0.157843	0.315187	0.471456	0.625886	0.777323	0.923778	1.123611	1.228632
0.2	0.142749	0.284598	0.424515	0.561159	0.692565	0.815477	0.968414	1.023680
0.3	0.129099	0.256978	0.382255	0.503165	0.617194	0.720366	0.837712	0.868832
0.4	0.116754	0.232040	0.344209	0.451198	0.550154	0.636769	0.727031	0.745203
0.5	0.105589	0.209522	0.309957	0.404629	0.490508	0.563236	0.632830	0.643694
0.6	0.095492	0.189191	0.279118	0.362893	0.437428	0.498504	0.552287	0.558890
0.7	0.086361	0.170833	0.251353	0.325486	0.390178	0.441478	0.483134	0.487198
0.8	0.078103	0.154256	0.226354	0.291957	0.348109	0.391204	0.423535	0.426062
0.9	0.070634	0.139289	0.203845	0.261901	0.310642	0.346851	0.371996	0.373579
1.0	0.063880	0.125775	0.183579	0.234956	0.277267	0.307694	0.327288	0.328286
1.2	0.052247	0.102553	0.148899	0.189138	0.221027	0.242523	0.254485	0.254889
1.4	0.042733	0.083620	0.120780	0.152298	0.176336	0.191533	0.198885	0.199051
1.6	0.034951	0.068183	0.097979	0.122667	0.140792	0.151541	0.156087	0.156156
1.8	0.028587	0.055597	0.079488	0.098829	0.112497	0.120105	0.122932	0.122961
2.0	0.023381	0.045335	0.064492	0.079644	0.089954	0.095342	0.097108	0.097121
2.2	0.019123	0.036967	0.052329	0.064201	0.071979	0.075797	0.076905	0.076911
2.4	0.015641	0.030145	0.042463	0.051766	0.057635	0.060342	0.061040	0.061043
2.6	0.012793	0.024582	0.034460	0.041750	0.046179	0.048100	0.048541	0.048542
2.8	0.010463	0.020045	0.027968	0.033680	0.037024	0.038387	0.038667	0.038668
3.0	0.008558	0.016347	0.022700	0.027177	0.029702	0.030670	0.030848	0.030848
3.5	0.005178	0.009817	0.013477	0.015912	0.017164	0.017576	0.017634	0.017634
4.0	0.003132	0.005896	0.008005	0.009330	0.009951	0.010128	0.010147	0.010147
4.5	0.001895	0.003542	0.004756	0.005478	0.005787	0.005862	0.005869	0.005869
5.0	0.001147	0.002127	0.002828	0.003221	0.003374	0.003407	0.003409	0.003409
5.5	0.000694	0.001278	0.001682	0.001896	0.001972	0.001986	0.001987	0.001987
6.0	0.000420	0.000768	0.001001	0.001117	0.001155	0.001162	0.001162	0.001162
6.5	0.000254	0.000461	0.000596	0.000659	0.000678	0.000681	0.000681	0.000681
7.0	0.000154	0.000277	0.000355	0.000389	0.000399	0.000400	0.000400	0.000400
7.5	0.000093	0.000167	0.000211	0.000230	0.000235	0.000235	0.000235	0.000235
8.0	0.000056	0.000100	0.000126	0.000136	0.000139	0.000139	0.000139	0.000139
8.5	0.000034	0.000060	0.000075	0.000081	0.000082	0.000082	0.000082	0.000082
9.0	0.000021	0.000036	0.000045	0.000048	0.000048	0.000048	0.000048	0.000048
9.5	0.000012	0.000022	0.000027	0.000028	0.000029	0.000029	0.000029	0.000029
10.0	0.000008	0.000013	0.000016	0.000017	0.000017	0.000017	0.000017	0.000017

$$\left[\begin{smallmatrix} (-3)2 \\ 6 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)5 \\ 6 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)8 \\ 6 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-3)1 \\ 7 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-3)1 \\ 7 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-3)2 \\ 7 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-3)4 \\ 7 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-2)2 \\ 11 \end{smallmatrix} \right]$$

$$27.5. \quad f_m(x) = \int_0^\infty t^m e^{-t^2 - \frac{x}{t}} dt \quad \text{and}$$

Related Integrals

$$m=0, 1, 2, \dots$$

Differential Equations

$$27.5.1 \quad x f_m''' - (m-1) f_m'' + 2 f_m = 0$$

$$27.5.2 \quad f_m' = -f_{m-1} \quad (m=1, 2, \dots)$$

Recurrence Relation

$$27.5.3 \quad 2 f_m = (m-1) f_{m-2} + x f_{m-3} \quad (m \geq 3)$$

Power Series Representations

$$27.5.4 \quad 2 f_1(x) = \sum_{k=0}^{\infty} (a_k \ln x + b_k) x^k$$

$$a_k = \frac{-2a_{k-2}}{k(k-1)(k-2)} \quad b_k = \frac{-2b_{k-2} - (3k^2 - 6k + 2)a_k}{k(k-1)(k-2)}$$

$$a_0 = a_1 = 0$$

$$a_2 = -b_0$$

$$b_0 = 1$$

$$b_1 = -\sqrt{\pi}$$

$$b_2 = \frac{3}{2}(1-\gamma)$$

(For γ , see chapter 6.)

27.5.5

$$2f_1(x) = 1 - \sqrt{\pi}x + .6342x^2 + .5908x^3 - .1431x^4 \\ - .01968x^5 + .00324x^6 + .000188x^7 \dots \\ - x^2 \ln x(1 - .08333x^2 + .001389x^4 - .0000083x^6 + \dots)$$

27.5.6

$$2f_2(x) = \frac{\sqrt{\pi}}{2} - x + \frac{\sqrt{\pi}}{2}x^2 - .3225x^3 - .1477x^4 + .03195x^5 \\ + .00328x^6 - .000491x^7 - .0000235x^8 \dots \\ + x^3 \ln x(\frac{1}{3} - .01667x^2 + .000198x^4 - \dots)$$

27.5.7

$$2f_3(x) = 1 - \frac{\sqrt{\pi}}{2}x + \frac{x^2}{2} - .2954x^3 + .1014x^4 + .02954x^5 \\ - .00578x^6 - .00047x^7 + .000064x^8 \dots \\ - x^4 \ln x(.0833 - .00278x^2 + .000025x^4 - \dots)$$

Asymptotic Representation

27.5.8

$$f_m(x) \sim \sqrt{\frac{\pi}{3}} 3^{-\frac{m}{2}} v^{\frac{m}{2}} e^{-v} \left(a_0 + \frac{a_1}{v} + \frac{a_2}{v^2} + \dots + \frac{a_k}{v^k} + \dots \right) \\ (x \rightarrow \infty)$$

$$v = 3 \left(\frac{x}{2} \right)^{2/3}$$

$$a_0 = 1, a_1 = \frac{1}{12} (3m^2 + 3m - 1)$$

$$12(k+2)a_{k+2} = -(12k^2 + 36k - 3m^2 - 3m + 25)a_{k+1} \\ + \frac{1}{2}(m-2k)(2k+3-m)(2k+3+2m)a_k \\ (k=0, 1, 2 \dots)$$

$$27.5.9 \quad g_1(x) + ig_2(x) = \int_0^\infty t^3 e^{-t^2 + i \frac{x}{t}} dt$$

27.5.10

$$g_1(x) = \mathcal{R}f_3(ix) \quad g_2(x) = -\mathcal{I}f_3(ix)$$

Asymptotic Representation

27.5.11

$$g_1(x) = \left(\frac{\pi}{3} \right)^{1/2} \frac{x}{2} \exp \left[-\frac{3}{2} \left(\frac{x}{2} \right)^{2/3} \right] (A \sin \theta + B \cos \theta)$$

27.5.12

$$g_2(x) = -\left(\frac{\pi}{3} \right)^{1/2} \frac{x}{2} \exp \left[-\frac{3}{2} \left(\frac{x}{2} \right)^{2/3} \right] (A \cos \theta - B \sin \theta)$$

$$\theta = \frac{3}{2} \sqrt{3} \left(\frac{x}{2} \right)^{2/3}$$

$$A \sim a_0 - a_3 \left(\frac{2}{x} \right)^2 + \frac{1}{2} \left[a_1 \left(\frac{2}{x} \right)^{2/3} - a_2 \left(\frac{2}{x} \right)^{4/3} \right. \\ \left. - a_4 \left(\frac{2}{x} \right)^{8/3} + a_5 \left(\frac{2}{x} \right)^{10/3} - \dots \right] \quad (x \rightarrow \infty)$$

$$B \sim \sqrt{3} \left[a_1 \left(\frac{2}{x} \right)^{2/3} + a_2 \left(\frac{2}{x} \right)^{4/3} - a_4 \left(\frac{2}{x} \right)^{8/3} \right. \\ \left. - a_5 \left(\frac{2}{x} \right)^{10/3} + \dots \right] \quad (x \rightarrow \infty)$$

$$a_0 = 1 \quad a_1 = .972222 \quad a_2 = .148534$$

$$a_3 = -.017879 \quad a_4 = .004594 \quad a_5 = -.000762$$

[27.7] M. Abramowitz, Evaluation of the integral $\int_0^\infty e^{-u^2 - z/u} du$, J. Math. Phys. **32**, 188-192 (1953).

[27.8] H. Faxén, Expansion in series of the integral $\int_0^\infty \exp[-x(t \pm t^{-n})] t^2 dt$, Ark. Mat., Astr., Fys. **15**, 13, 1-57 (1921).

[27.9] J. E. Kilpatrick and M. F. Kilpatrick, Discrete energy levels associated with the Lennard-Jones potential, J. Chem. Phys. **19**, 7, 930-933 (1951).

[27.10] U. E. Kruse and N. F. Ramsey, The integral $\int_0^\infty y^4 \exp(-y^2 + i \frac{x}{y}) dy$, J. Math. Phys. **30**, 40 (1951).

[27.11] O. Laporte, Absorption coefficients for thermal neutrons, Phys. Rev. **52**, 72-74 (1937).

[27.12] H. C. Torrey, Notes on intensities of radio frequency spectra, Phys. Rev. **59**, 293 (1941).

[27.13] C. T. Zahn, Absorption coefficients for thermal neutrons, Phys. Rev. **52**, 67-71 (1937). $\int_0^\infty y^n e^{-y^2 - x/\sqrt{y}} dy$ for $n=0, \frac{1}{2}, 1$; $x=0(.01).1(.1)1$.

$$f_n(x) = \int_0^{\infty} t^n e^{-t^2 - \frac{x}{t}} dt$$

Table 27.5

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	x	$f_1(x)$	$f_2(x)$	$f_3(x)$	x	$f_1(x)$	$f_2(x)$	$f_3(x)$
0.00	0.5000	0.4431	0.5000	0.1	0.4263	0.3970	0.4580	0.6	0.2255	0.2415	0.3025
0.01	0.4914	0.4382	0.4956	0.2	0.3697	0.3573	0.4204	0.7	0.2015	0.2202	0.2793
0.02	0.4832	0.4333	0.4912	0.3	0.3238	0.3227	0.3864	0.8	0.1807	0.2011	0.2584
0.03	0.4753	0.4285	0.4869	0.4	0.2855	0.2923	0.3557	0.9	0.1626	0.1839	0.2392
0.04	0.4676	0.4238	0.4826	0.5	0.2531	0.2654	0.3278	1.0	0.1466	0.1685	0.2215
0.05	0.4602	0.4191	0.4784								

$$\left[\begin{smallmatrix} (-5)5 \\ 2 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-5)5 \\ 2 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-5)5 \\ 2 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-3)1 \\ 4 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-4)7 \\ 3 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-4)5 \\ 3 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)6 \\ 3 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-4)4 \\ 3 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-4)4 \\ 3 \end{smallmatrix} \right]$$

x	$\mathcal{R}_{f_3}(ix)$	$-\mathcal{I}_{f_3}(ix)$	x	$\mathcal{R}_{f_3}(ix)$	$-\mathcal{I}_{f_3}(ix)$	x	$\mathcal{R}_{f_3}(ix)$	$-\mathcal{I}_{f_3}(ix)$
0.0	0.50000	0.00000	4.0	-0.2626	0.0430	8.0	0.06078	-0.09808
0.2	0.49019	0.08754	4.2	-0.2552	+0.0094	8.5	0.07562	-0.07131
0.4	0.46229	0.16933	4.4	-0.2441	-0.0214	9.0	0.08221	-0.04496
0.6	0.41950	0.24139	4.6	-0.2299	-0.0490	9.5	0.08191	-0.02082
0.8	0.36543	0.30136	4.8	-0.2132	-0.0734	10.0	0.07626	-0.00010
1.0	0.30366	0.34805	5.0	-0.1945	-0.0944	10.5	0.06684	+0.01654
1.2	0.23746	0.38122	5.2	-0.1745	-0.1120	11.0	0.05507	0.02889
1.4	0.16972	0.40127	5.4	-0.1536	-0.1263	11.5	0.04224	0.03707
1.6	0.10288	0.40910	5.6	-0.1322	-0.1374	12.0	0.02937	0.04146
1.8	+0.03892	0.40592	5.8	-0.1108	-0.1455	12.5	0.01727	0.04259
2.0	-0.02062	0.39314	6.0	-0.0896	-0.1507	13.0	+0.00650	0.04109
2.2	-0.0746	0.3722	6.2	-0.0691	-0.1533	13.5	-0.00259	0.03758
2.4	-0.1221	0.3448	6.4	-0.0493	-0.1535	14.0	-0.00982	0.03268
2.6	-0.1629	0.3122	6.6	-0.0307	-0.1515	14.5	-0.01517	0.02696
2.8	-0.1966	0.2759	6.8	-0.0132	-0.1476	15.0	-0.01872	0.02089
3.0	-0.2233	0.2371	7.0	+0.00286	-0.14211	16.0	-0.02118	+0.00921
3.2	-0.2432	0.1971	7.2	0.01749	-0.13518	17.0	-0.01906	-0.00022
3.4	-0.2565	0.1569	7.4	0.03061	-0.12709	18.0	-0.01435	-0.00650
3.6	-0.2639	0.1173	7.6	0.04220	-0.11805	19.0	-0.00879	-0.00965
3.8	-0.2657	0.0792	7.8	0.05224	-0.10830	20.0	-0.00360	-0.01021

$$\left[\begin{smallmatrix} (-3)2 \\ 6 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-3)2 \\ 5 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)5 \\ 3 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-4)4 \\ 4 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-3)1 \\ 5 \end{smallmatrix} \right] \left[\begin{smallmatrix} (-4)7 \\ 5 \end{smallmatrix} \right]$$

Compiled from U. E. Kruse and N. F. Ramsey, The integral $\int_0^{\infty} y^2 \exp\left(-y^2 + i\frac{x}{y}\right) dy$, J. Math. Phys. 30, 40 (1951) (with permission).

$$27.6. f(x) = \int_0^{\infty} \frac{e^{-t^2}}{t+x} dt$$

Power Series Representation

27.6.1

$$f(x) = -e^{-x^2} \ln x + e^{-x^2} \left[\sqrt{\pi} \sum_{k=0}^{\infty} \frac{x^{2k+1}}{k!(2k+1)} - \sum_{k=1}^{\infty} \frac{x^{2k}}{k! 2k} - \frac{\gamma}{2} \right]$$

27.6.2

$$= -e^{-x^2} \ln x + \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k \psi(k+1) x^{2k}}{k!} + \sqrt{\pi} \sum_{k=0}^{\infty} \frac{(-2)^k x^{2k+1}}{1 \cdot 3 \cdot 5 \cdots (2k+1)}$$

(For γ and the digamma function $\psi(x)$, see chapter 6.)

Relation to the Exponential Integral

$$27.6.3 f(x) = -\frac{1}{2} e^{-x^2} \text{Ei}(x^2) + \sqrt{\pi} e^{-x^2} \int_0^x e^{t^2} dt$$

(For Ei(x) see chapter 5; $e^{-x^2} \int_0^x e^{t^2} dt$, see chapter 7.)

Asymptotic Representation

27.6.4

$$f(x) \sim \frac{\sqrt{\pi}}{2} \left[\frac{1}{x} + \frac{1}{2x^3} + \frac{1 \cdot 3}{4x^5} + \frac{1 \cdot 3 \cdot 5}{8x^7} + \dots \right] - \frac{1}{2} \left[\frac{1}{x^2} + \frac{1}{x^4} + \frac{2!}{x^6} + \frac{3!}{x^8} + \dots \right] \quad (x \rightarrow \infty)$$

[27.14] A. Erdélyi, Note on the paper "On a definite integral" by R. H. Ritchie, Math. Tables Aids Comp. 4, 31, 179 (1950).

[27.15] E. T. Goodwin and J. Staton, Table of $\int_0^{\infty} \frac{e^{-u^2}}{u+x} du$, Quart. J. Mech. Appl. Math. 1, 319 (1948). $x=0(.02)2(.05)3(.1)10$. Auxiliary function for $x=0(.01)1$.

[27.16] R. H. Ritchie, On a definite integral, Math. Tables Aids Comp. 4, 30, 75 (1950).

Table 27.6

$$f(x) = \int_0^{\infty} \frac{e^{-t^2}}{t+x} dt$$

x	$f(x) + \ln x$	x	$f(x) + \ln x$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0.00	-0.2886	0.50	0.2704	1.0	0.6051	2.0	0.3543	3.0	0.2519
0.05	-0.2081	0.55	0.3100	1.1	0.5644	2.1	0.3404	3.5	0.2203
0.10	-0.1375	0.60	0.3479	1.2	0.5291	2.2	0.3276	4.0	0.1958
0.15	-0.0735	0.65	0.3842	1.3	0.4980	2.3	0.3157	4.5	0.1762
0.20	-0.0146	0.70	0.4192	1.4	0.4705	2.4	0.3046	5.0	0.1602
0.25	+0.0402	0.75	0.4529	1.5	0.4460	2.5	0.2944	5.5	0.1468
0.30	0.0915	0.80	0.4854	1.6	0.4239	2.6	0.2848	6.0	0.1356
0.35	0.1398	0.85	0.5168	1.7	0.4040	2.7	0.2758	6.5	0.1259
0.40	0.1856	0.90	0.5472	1.8	0.3860	2.8	0.2673	7.0	0.1175
0.45	0.2290	0.95	0.5766	1.9	0.3695	2.9	0.2594	7.5	0.1102
0.50	0.2704	1.00	0.6051	2.0	0.3543	3.0	0.2519	8.0	0.1037
$\left[\begin{smallmatrix} (-3)1 \\ 4 \end{smallmatrix} \right]$		$\left[\begin{smallmatrix} (-4)2 \\ 3 \end{smallmatrix} \right]$		$\left[\begin{smallmatrix} (-4)7 \\ 4 \end{smallmatrix} \right]$		$\left[\begin{smallmatrix} (-4)1 \\ 3 \end{smallmatrix} \right]$		$\left[\begin{smallmatrix} (-4)9 \\ 4 \end{smallmatrix} \right]$	

Compiled from E. T. Goodwin and J. Staton, Table of $\int_0^{\infty} \frac{e^{-u^2}}{u+x} du$, Quart. J. Mech. Appl. Math. 1, 319 (1948) (with permission).

27.7. Dilogarithm

(Spence's Integral for $n=2$)

$$27.7.1 \quad f(x) = -\int_1^x \frac{\ln t}{t-1} dt$$

Series Expansion

$$27.7.2 \quad f(x) = \sum_{k=1}^{\infty} (-1)^k \frac{(x-1)^k}{k^2} \quad (2 \geq x \geq 0)$$

Functional Relationships

27.7.3

$$f(x) + f(1-x) = -\ln x \ln(1-x) + \frac{\pi^2}{6} \quad (1 \geq x \geq 0)$$

27.7.4

$$f(1-x) + f(1+x) = \frac{1}{2} f(1-x^2) \quad (1 \geq x > 0)$$

$$27.7.5 \quad f(x) + f\left(\frac{1}{x}\right) = -\frac{1}{2} (\ln x)^2 \quad (0 \leq x \leq 1)$$

27.7.6

$$f(x+1) - f(x) = -\ln x \ln(x+1) - \frac{\pi^2}{12} - \frac{1}{2} f(x^2) \quad (2 \geq x \geq 0)$$

Relation to Debye Functions

$$27.7.7 \quad f(e^{-t}) = -f(e^t) - \frac{t^2}{2} = \int_0^t \frac{t dt}{e^t - 1}$$

[27.17] L. Lewin, Dilogarithms and associated functions (Macdonald, London, England, 1958).

[27.18] K. Mitchell, Tables of the function $\int_0^x \frac{-\log|1-y|}{y} dy$, with an account of some properties of this and related functions, Phil. Mag. 40, 351-368 (1949). $x = -1(.01)1$; $x = 0(.001).5$, 9D.

[27.19] E. O. Powell, An integral related to the radiation integrals, Phil. Mag. 7, 34, 600-607 (1943). $\int_1^x \frac{\log y}{y-1} dy$, $x = 0(.01)2(.02)6$, 7D.

[27.20] A. van Wijngaarden, Polylogarithms, by the Staff of the Computation Department, Report R24, Mathematisch Centrum, Amsterdam, Holland (1954). $F_n(z) = \sum_{k=1}^{\infty} k^{-n} z^k$ for $z = x = -1(.01)1$; $z = ix$, for $x = 0(.01)1$; $z = e^{i\pi\alpha/2}$ for $\alpha = 0(.01)2$, 10D.

Dilogarithm

Table 27.7

$$f(x) = -\int_1^x \frac{\ln t}{t-1} dt$$

x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0.00	1.64493 4067	0.10	1.29971 4723	0.20	1.07479 4600	0.30	0.88937 7624	0.40	0.72758 6308
0.01	1.58862 5448	0.11	1.27452 9160	0.21	1.05485 9830	0.31	0.87229 1733	0.41	0.71239 5042
0.02	1.54579 9712	0.12	1.25008 7584	0.22	1.03527 7934	0.32	0.85542 7404	0.42	0.69736 1058
0.03	1.50789 9041	0.13	1.22632 0101	0.23	1.01603 0062	0.33	0.83877 6261	0.43	0.68247 9725
0.04	1.47312 5860	0.14	1.20316 7961	0.24	0.99709 9088	0.34	0.82233 0471	0.44	0.66774 6644
0.05	1.44063 3797	0.15	1.18058 1124	0.25	0.97846 9393	0.35	0.80608 2689	0.45	0.65315 7631
0.06	1.40992 8300	0.16	1.15851 6487	0.26	0.96012 6675	0.36	0.79002 8024	0.46	0.63870 8705
0.07	1.38068 5041	0.17	1.13693 6560	0.27	0.94205 7798	0.37	0.77415 3992	0.47	0.62439 6071
0.08	1.35267 5161	0.18	1.11580 8451	0.28	0.92425 0654	0.38	0.75846 0483	0.48	0.61021 6108
0.09	1.32572 8728	0.19	1.09510 3088	0.29	0.90669 4053	0.39	0.74293 9737	0.49	0.59616 5361
0.10	1.29971 4723	0.20	1.07479 4600	0.30	0.88937 7624	0.40	0.72758 6308	0.50	0.58224 0526

 $\left[\begin{smallmatrix} (-3) \\ 2 \end{smallmatrix} \right]$
 $\left[\begin{smallmatrix} (-4) \\ 1 \end{smallmatrix} \right]$
 $\left[\begin{smallmatrix} (-5) \\ 7 \end{smallmatrix} \right]$
 $\left[\begin{smallmatrix} (-5) \\ 6 \end{smallmatrix} \right]$
 $\left[\begin{smallmatrix} (-5) \\ 5 \end{smallmatrix} \right]$

From K. Mitchell, Tables of the function $\int_0^x \frac{z - \log |1-y|}{y} dy$, with an account of some properties of this and related functions, Phil. Mag. 40, 351-368 (1949) (with permission).

27.8. Clausen's Integral and Related Summations

27.8.1

$$f(\theta) = -\int_0^\theta \ln \left(2 \sin \frac{t}{2} \right) dt = \sum_{k=1}^{\infty} \frac{\sin k\theta}{k^2} \quad (0 \leq \theta \leq \pi)$$

Series Representation

27.8.2

$$f(\theta) = -\theta \ln |\theta| + \theta + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k)!} B_{2k} \frac{\theta^{2k+1}}{2k(2k+1)} \quad \left(0 \leq \theta < \frac{\pi}{2} \right)$$

27.8.3

$$f(\pi - \theta) = \theta \ln 2 - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k)!} B_{2k} (2^{2k} - 1) \frac{\theta^{2k+1}}{2k(2k+1)} \quad (\pi/2 < \theta < \pi)$$

Functional Relationship

$$27.8.4 \quad f(\pi - \theta) = f(\theta) - \frac{1}{2} f(2\theta) \quad \left(0 \leq \theta \leq \frac{\pi}{2} \right)$$

Relation to Spence's Integral

27.8.5

$$if(\theta) = g(e^{i\theta}) + \frac{\theta^2}{4} \text{ where } g(x) = \int_1^x \frac{dt}{t} \ln |1+t|$$

Summable Series

27.8.6

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} = -\ln \left(2 \sin \frac{\theta}{2} \right) \quad (0 < \theta < 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^2} = \frac{\pi^2}{6} - \frac{\pi\theta}{2} + \frac{\theta^2}{4} \quad (0 \leq \theta \leq 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n^4} = \frac{\pi^4}{90} - \frac{\pi^2\theta^2}{12} + \frac{\pi\theta^3}{12} - \frac{\theta^4}{48} \quad (0 \leq \theta \leq 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n} = \frac{1}{2} (\pi - \theta) \quad (0 < \theta < 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^3} = \frac{\pi^2\theta}{6} - \frac{\pi\theta^2}{4} + \frac{\theta^3}{12} \quad (0 \leq \theta \leq 2\pi)$$

$$\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^5} = \frac{\pi^4\theta}{90} - \frac{\pi^2\theta^3}{36} + \frac{\pi\theta^4}{48} - \frac{\theta^5}{240} \quad (0 \leq \theta \leq 2\pi)$$

[27.21] A. Ashour and A. Sabri, Tabulation of the function

$$\psi(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}, \text{ Math. Tables Aids Comp. 10, 54, 57-65 (1956).}$$

[27.22] T. Clausen, Über die Zerlegung reeller gebrochener Funktionen, J. Reine Angew. Math. 8, 298-300 (1832). $x = 0^\circ(1^\circ)180^\circ$, 16D.

[27.23] L. B. W. Jolley, Summation of series (Chapman Publishing Co., London, England, 1925).

[27.24] A. D. Wheelon, A short table of summable series, Report No. SM-14642, Douglas Aircraft Co., Inc., Santa Monica, Calif. (1953).

Table 27.8

Clausen's Integral

$$f(\theta) = -\int_0^\theta \ln \left(2 \sin \frac{t}{2} \right) dt$$

θ°	$f(\theta) + \theta \ln \theta$	θ°	$f(\theta)$	θ°	$f(\theta)$	θ°	$f(\theta)$	θ°	$f(\theta)$
0	0.000000	15	0.612906	30	0.864379	60	1.014942	90	0.915966
1	0.017453	16	0.635781	32	0.886253	62	1.014421	95	0.883872
2	0.034908	17	0.657571	34	0.906001	64	1.012886	100	0.848287
3	0.052362	18	0.678341	36	0.923755	66	1.010376	105	0.809505
4	0.069818	19	0.698149	38	0.939633	68	1.006928	110	0.767800
5	0.087276	20	0.717047	40	0.953741	70	1.002576	115	0.723427
6	0.104735	21	0.735080	42	0.966174	72	0.997355	120	0.676628
7	0.122199	22	0.752292	44	0.977020	74	0.991294	125	0.627629
8	0.139664	23	0.768719	46	0.986357	76	0.984425	130	0.576647
9	0.157133	24	0.784398	48	0.994258	78	0.976776	135	0.523889
10	0.174607	25	0.799360	50	1.000791	80	0.968375	140	0.469554
11	0.192084	26	0.813635	52	1.006016	82	0.959247	145	0.413831
12	0.209567	27	0.827249	54	1.009992	84	0.949419	150	0.356908
13	0.227055	28	0.840230	56	1.012773	86	0.938914	160	0.240176
14	0.244549	29	0.852599	58	1.014407	88	0.927755	170	0.120755
15	0.262049	30	0.864379	60	1.014942	90	0.915966	180	0.000000

$$\left[\begin{smallmatrix} (-7)8 \\ 3 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)1 \\ 4 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)3 \\ 4 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)1 \\ 4 \end{smallmatrix} \right]$$

$$\left[\begin{smallmatrix} (-4)4 \\ 6 \end{smallmatrix} \right]$$

Compiled from A. Ashour and A. Sabri, Tabulation of the function $\psi(\theta) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^3}$, Math. Tables Aids Comp. 10, 54, 57-65 (1956) (with permission).

27.9. Vector-Addition Coefficients

(Wigner coefficients or Clebsch-Gordan coefficients)

Definition

27.9.1

$$(j_1 j_2 m_1 m_2 | j_1 j_2 j m) = \delta(m, m_1 + m_2) \cdot \sqrt{\frac{(j_1 + j_2 - j)!(j + j_1 - j_2)!(j + j_2 - j_1)!(2j + 1)}{(j + j_1 + j_2 + 1)!}}$$

$$\cdot \sum_k \frac{(-1)^k \sqrt{(j_1 + m_1)!(j_1 - m_1)!(j_2 + m_2)!(j_2 - m_2)!(j + m)!(j - m)!}}{k!(j_1 + j_2 - j - k)!(j_1 - m_1 - k)!(j_2 + m_2 - k)!(j - j_2 + m_1 + k)!(j - j_1 - m_2 + k)!}$$

$$\delta(i, k) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

Conditions

$$27.9.2 \quad j_1, j_2, j = +n \text{ or } +\frac{n}{2} \quad (n = \text{integer})$$

$$27.9.3 \quad j_1 + j_2 + j = n$$

$$\left. \begin{aligned} 27.9.4 \quad j_1 + j_2 - j \\ 27.9.5 \quad j_1 - j_2 + j \\ 27.9.6 \quad -j_1 + j_2 + j \end{aligned} \right\} \geq 0$$

$$27.9.7 \quad m_1, m_2, m = \pm n \text{ or } \pm \frac{n}{2}$$

$$27.9.8 \quad |m_1| \leq j_1, |m_2| \leq j_2, |m| \leq j$$

$$27.9.9 \quad (j_1 j_2 m_1 m_2 | j_1 j_2 j m) = 0 \quad m_1 + m_2 \neq m$$

Special Values

$$27.9.10 \quad (j_1 0 m_1 0 | j_1 0 j m) = \delta(j_1, j) \delta(m_1, m)$$

$$27.9.11 \quad (j_1 j_2 0 0 | j_1 j_2 j 0) = 0 \quad j_1 + j_2 + j = 2n + 1$$

$$27.9.12 \quad (j_1 j_1 m_1 m_1 | j_1 j_1 j m) = 0 \quad 2j_1 + j = 2n + 1$$

Symmetry Relations

27.9.13

$$(j_1 j_2 m_1 m_2 | j_1 j_2 j m) \\ = (-1)^{j_1+j_2-j} (j_1 j_2 - m_1 - m_2 | j_1 j_2 j - m)$$

27.9.14

$$= (j_2 j_1 - m_2 - m_1 | j_2 j_1 j - m)$$

27.9.15

$$= (-1)^{j_1+j_2-j} (j_2 j_1 m_1 m_2 | j_2 j_1 j m)$$

27.9.16

$$= \sqrt{\frac{2j+1}{2j_1+1}} (-1)^{j_1+m_2} (j j_2 - m m_2 \\ | j j_2 j_1 - m_1)$$

27.9.17

$$= \sqrt{\frac{2j+1}{2j_1+1}} (-1)^{j_1-m_1+j-m} (j j_2 m - m_2 \\ | j j_2 j_1 m_1)$$

27.9.18

$$= \sqrt{\frac{2j+1}{2j_1+1}} (-1)^{j-m+j_1-m_1} (j_2 j m_2 - m \\ | j_2 j j_1 - m_1)$$

27.9.19

$$= \sqrt{\frac{2j+1}{2j_2+1}} (-1)^{j_1-m_1} (j_1 j m_1 - m \\ | j_1 j j_2 - m_2)$$

27.9.20

$$= \sqrt{\frac{2j+1}{2j_2+1}} (-1)^{j_1-m_1} (j j_1 m - m_1 \\ | j j_1 j_2 m_2)$$

 $(j_1 \frac{1}{2} m_1 m_2 | j_1 \frac{1}{2} j m)$

Table 27.9.1

$j =$	$m_2 = \frac{1}{2}$	$m_2 = -\frac{1}{2}$
$j_1 + \frac{1}{2}$	$\sqrt{\frac{j_1+m+\frac{1}{2}}{2j_1+1}}$	$\sqrt{\frac{j_1-m+\frac{1}{2}}{2j_1+1}}$
$j_1 - \frac{1}{2}$	$-\sqrt{\frac{j_1-m+\frac{1}{2}}{2j_1+1}}$	$\sqrt{\frac{j_1+m+\frac{1}{2}}{2j_1+1}}$

 $(j_1 1 m_1 m_2 | j_1 1 j m)$

Table 27.9.2

$j =$	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$
$j_1 + 1$	$\sqrt{\frac{(j_1+m)(j_1+m+1)}{(2j_1+1)(2j_1+2)}}$	$\sqrt{\frac{(j_1-m+1)(j_1+m+1)}{(2j_1+1)(j_1+1)}}$	$\sqrt{\frac{(j_1-m)(j_1-m+1)}{(2j_1+1)(2j_1+2)}}$
j_1	$-\sqrt{\frac{(j_1+m)(j_1-m+1)}{2j_1(j_1+1)}}$	$\frac{m}{\sqrt{j_1(j_1+1)}}$	$\sqrt{\frac{(j_1-m)(j_1+m+1)}{2j_1(j_1+1)}}$
$j_1 - 1$	$\sqrt{\frac{(j_1-m)(j_1-m+1)}{2j_1(2j_1+1)}}$	$-\sqrt{\frac{(j_1-m)(j_1+m)}{j_1(2j_1+1)}}$	$\sqrt{\frac{(j_1+m+1)(j_1+m)}{2j_1(2j_1+1)}}$

Table 27.9.3

 $(j_1 \frac{1}{2} m_1 m_2 | j_1 \frac{1}{2} j m)$

$j =$	$m_2 = \frac{1}{2}$	$m_2 = \frac{1}{2}$
$j_1 + \frac{1}{2}$	$\sqrt{\frac{(j_1 + m - \frac{1}{2})(j_1 + m + \frac{1}{2})(j_1 + m + \frac{3}{2})}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)}}$	$\sqrt{\frac{3(j_1 + m + \frac{1}{2})(j_1 + m + \frac{3}{2})(j_1 - m + \frac{3}{2})}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)}}$
$j_1 + \frac{1}{2}$	$-\sqrt{\frac{3(j_1 + m - \frac{1}{2})(j_1 + m + \frac{1}{2})(j_1 - m + \frac{3}{2})}{2j_1(2j_1 + 1)(2j_1 + 3)}}$	$-(j_1 - 3m + \frac{1}{2})\sqrt{\frac{j_1 + m + \frac{1}{2}}{2j_1(2j_1 + 1)(2j_1 + 3)}}$
$j_1 - \frac{1}{2}$	$\sqrt{\frac{3(j_1 + m - \frac{1}{2})(j_1 - m + \frac{1}{2})(j_1 - m + \frac{3}{2})}{(2j_1 - 1)(2j_1 + 1)(2j_1 + 2)}}$	$-(j_1 + 3m - \frac{1}{2})\sqrt{\frac{j_1 - m + \frac{1}{2}}{(2j_1 - 1)(2j_1 + 1)(2j_1 + 2)}}$
$j_1 - \frac{1}{2}$	$-\sqrt{\frac{(j_1 - m - \frac{1}{2})(j_1 - m + \frac{1}{2})(j_1 - m + \frac{3}{2})}{2j_1(2j_1 - 1)(2j_1 + 1)}}$	$\sqrt{\frac{3(j_1 + m - \frac{1}{2})(j_1 - m - \frac{1}{2})(j_1 - m + \frac{1}{2})}{2j_1(2j_1 - 1)(2j_1 + 1)}}$
$j =$	$m_2 = -\frac{1}{2}$	$m_2 = -\frac{1}{2}$
$j_1 + \frac{1}{2}$	$\sqrt{\frac{3(j_1 + m + \frac{3}{2})(j_1 - m + \frac{1}{2})(j_1 - m + \frac{3}{2})}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)}}$	$\sqrt{\frac{(j_1 - m - \frac{1}{2})(j_1 - m + \frac{1}{2})(j_1 - m + \frac{3}{2})}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)}}$
$j_1 + \frac{1}{2}$	$(j_1 + 3m + \frac{1}{2})\sqrt{\frac{j_1 - m + \frac{1}{2}}{2j_1(2j_1 + 1)(2j_1 + 3)}}$	$\sqrt{\frac{3(j_1 + m + \frac{3}{2})(j_1 - m - \frac{1}{2})(j_1 - m + \frac{1}{2})}{2j_1(2j_1 + 1)(2j_1 + 3)}}$
$j_1 - \frac{1}{2}$	$-(j_1 - 3m - \frac{1}{2})\sqrt{\frac{j_1 + m + \frac{1}{2}}{(2j_1 - 1)(2j_1 + 1)(2j_1 + 2)}}$	$\sqrt{\frac{3(j_1 + m + \frac{1}{2})(j_1 + m + \frac{3}{2})(j_1 - m - \frac{1}{2})}{(2j_1 - 1)(2j_1 + 1)(2j_1 + 2)}}$
$j_1 - \frac{1}{2}$	$-\sqrt{\frac{3(j_1 + m - \frac{1}{2})(j_1 + m + \frac{1}{2})(j_1 - m - \frac{1}{2})}{2j_1(2j_1 - 1)(2j_1 + 1)}}$	$\sqrt{\frac{(j_1 + m - \frac{1}{2})(j_1 + m + \frac{1}{2})(j_1 + m + \frac{3}{2})}{2j_1(2j_1 - 1)(2j_1 + 1)}}$

Table 27.9.4

$(j_1 \ 2 \ m_1 \ m_2 \ | \ j_1 \ 2 \ j \ m)$

$j =$	$m_2 = 2$	$m_2 = 1$	$m_2 = 0$
$j_1 + 2$	$\sqrt{\frac{(j_1 + m - 1)(j_1 + m)(j_1 + m + 1)(j_1 + m + 2)}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)}}$	$\sqrt{\frac{(j_1 - m + 2)(j_1 + m + 2)(j_1 + m + 1)(j_1 + m)}{(2j_1 + 1)(j_1 + 1)(2j_1 + 3)(j_1 + 2)}}$	$\sqrt{\frac{3(j_1 - m + 2)(j_1 - m + 1)(j_1 + m + 2)(j_1 + m + 1)}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)(j_1 + 2)}}$
$j_1 + 1$	$-\sqrt{\frac{(j_1 + m - 1)(j_1 + m)(j_1 + m + 1)(j_1 - m + 2)}{2j_1(j_1 + 1)(j_1 + 2)(2j_1 + 1)}}$	$-(j_1 - 2m + 2)\sqrt{\frac{(j_1 + m + 1)(j_1 + m)}{2j_1(2j_1 + 1)(j_1 + 1)(j_1 + 2)}}$	$m\sqrt{\frac{3(j_1 - m + 1)(j_1 + m + 1)}{j_1(2j_1 + 1)(j_1 + 1)(j_1 + 2)}}$
j_1	$\sqrt{\frac{3(j_1 + m - 1)(j_1 + m)(j_1 - m + 1)(j_1 - m + 2)}{(2j_1 - 1)2j_1(j_1 + 1)(2j_1 + 3)}}$	$(1 - 2m)\sqrt{\frac{3(j_1 - m + 1)(j_1 + m)}{(2j_1 - 1)j_1(2j_1 + 2)(2j_1 + 3)}}$	$\frac{3m^2 - j_1(j_1 + 1)}{\sqrt{(2j_1 - 1)j_1(j_1 + 1)(2j_1 + 3)}}$
$j_1 - 1$	$-\sqrt{\frac{(j_1 + m - 1)(j_1 - m)(j_1 - m + 1)(j_1 - m + 2)}{2(j_1 - 1)j_1(j_1 + 1)(2j_1 + 1)}}$	$(j_1 + 2m - 1)\sqrt{\frac{(j_1 - m + 1)(j_1 - m)}{(j_1 - 1)j_1(2j_1 + 1)(2j_1 + 2)}}$	$-m\sqrt{\frac{3(j_1 - m)(j_1 + m)}{(j_1 - 1)j_1(2j_1 + 1)(j_1 + 1)}}$
$j_1 - 2$	$\sqrt{\frac{(j_1 - m - 1)(j_1 - m)(j_1 - m + 1)(j_1 - m + 2)}{(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)}}$	$-\sqrt{\frac{(j_1 - m + 1)(j_1 - m)(j_1 - m - 1)(j_1 + m - 1)}{(j_1 - 1)(2j_1 - 1)j_1(2j_1 + 1)}}$	$\sqrt{\frac{3(j_1 - m)(j_1 - m - 1)(j_1 + m)(j_1 + m - 1)}{(2j_1 - 2)(2j_1 - 1)j_1(2j_1 + 1)}}$
$j =$	$m_2 = -1$	$m_2 = -2$	
$j_1 + 2$	$\sqrt{\frac{(j_1 - m + 2)(j_1 - m + 1)(j_1 - m)(j_1 + m + 2)}{(2j_1 + 1)(j_1 + 1)(2j_1 + 3)(j_1 + 2)}}$	$\sqrt{\frac{(j_1 - m - 1)(j_1 - m)(j_1 - m + 1)(j_1 - m + 2)}{(2j_1 + 1)(2j_1 + 2)(2j_1 + 3)(2j_1 + 4)}}$	
$j_1 + 1$	$(j_1 + 2m + 2)\sqrt{\frac{(j_1 - m + 1)(j_1 - m)}{j_1(2j_1 + 1)(2j_1 + 2)(j_1 + 2)}}$	$\sqrt{\frac{(j_1 - m - 1)(j_1 - m)(j_1 - m + 1)(j_1 + m + 2)}{j_1(2j_1 + 1)(j_1 + 1)(2j_1 + 4)}}$	
j_1	$(2m + 1)\sqrt{\frac{3(j_1 - m)(j_1 + m + 1)}{(2j_1 - 1)j_1(2j_1 + 2)(2j_1 + 3)}}$	$\sqrt{\frac{3(j_1 - m - 1)(j_1 - m)(j_1 + m + 1)(j_1 + m + 2)}{(2j_1 - 1)j_1(2j_1 + 2)(2j_1 + 3)}}$	
$j_1 - 1$	$-(j_1 - 2m - 1)\sqrt{\frac{(j_1 + m + 1)(j_1 + m)}{(j_1 - 1)j_1(2j_1 + 1)(2j_1 + 2)}}$	$\sqrt{\frac{(j_1 - m - 1)(j_1 + m)(j_1 + m + 1)(j_1 + m + 2)}{(j_1 - 1)j_1(2j_1 + 1)(2j_1 + 2)}}$	
$j_1 - 2$	$-\sqrt{\frac{(j_1 - m - 1)(j_1 + m + 1)(j_1 + m)(j_1 + m - 1)}{(j_1 - 1)(2j_1 - 1)j_1(2j_1 + 1)}}$	$\sqrt{\frac{(j_1 + m - 1)(j_1 + m)(j_1 + m + 1)(j_1 + m + 2)}{(2j_1 - 2)(2j_1 - 1)2j_1(2j_1 + 1)}}$	

Table 27.9.5 [By use of symmetry relations, coefficients may be put in standard form $j_1 \leq j_2 \leq j$ and $m \geq 0$]

m_2	m	j_1	j	$(j_1 j_2 m_1 m_2 j_1 j_2 j m)$	
$j_2 = \frac{1}{2}$					
$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\sqrt{\frac{1}{2}}$	0. 70711
$\frac{1}{2}$	0	$\frac{1}{2}$	1	$\sqrt{\frac{1}{2}}$	0. 70711
$\frac{1}{2}$	1	$\frac{1}{2}$	1		1. 00000
$j_2 = 1$					
-1	0	1	1	$\sqrt{\frac{1}{2}}$	0. 70711
0	0	1	1		0. 00000
1	0	1	1	$-\sqrt{\frac{1}{2}}$	-0. 70711
0	1	1	1	$\sqrt{\frac{1}{2}}$	0. 70711
1	1	1	1	$-\sqrt{\frac{1}{2}}$	-0. 70711
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\sqrt{\frac{3}{2}}$	0. 81650
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\sqrt{\frac{1}{2}}$	0. 57735
1	$\frac{3}{2}$	$\frac{1}{2}$	$\frac{3}{2}$		1. 00000
-1	0	1	2	$\sqrt{\frac{1}{6}}$	0. 40825
0	0	1	2	$\sqrt{\frac{3}{2}}$	0. 81650
1	0	1	2	$\sqrt{\frac{1}{6}}$	0. 40825
0	1	1	2	$\sqrt{\frac{1}{2}}$	0. 70711
1	1	1	2	$\sqrt{\frac{1}{2}}$	0. 70711
1	2	1	2		1. 00000
$j_2 = \frac{3}{2}$					
$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$\sqrt{\frac{3}{10}}$	0. 73030
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$-\sqrt{\frac{1}{10}}$	-0. 25820
$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$	$-\sqrt{\frac{3}{2}}$	-0. 63246
$\frac{1}{2}$	$\frac{3}{2}$	1	$\frac{3}{2}$	$\sqrt{\frac{3}{2}}$	0. 63246
$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{3}{2}$	$-\sqrt{\frac{3}{2}}$	-0. 77460
-1	0	$\frac{1}{2}$	2	$\sqrt{\frac{1}{2}}$	0. 70711
$\frac{1}{2}$	0	$\frac{1}{2}$	2	$\sqrt{\frac{1}{2}}$	0. 70711
$\frac{3}{2}$	1	$\frac{1}{2}$	2	$\frac{1}{2}\sqrt{3}$	0. 86603
$\frac{5}{2}$	1	$\frac{1}{2}$	2		0. 50000
$\frac{3}{2}$	2	$\frac{1}{2}$	2		1. 00000
-1	0	$\frac{3}{2}$	2		0. 50000
$\frac{1}{2}$	0	$\frac{3}{2}$	2		0. 50000
$\frac{3}{2}$	0	$\frac{3}{2}$	2		-0. 50000
$\frac{5}{2}$	0	$\frac{3}{2}$	2		-0. 50000
-1	1	$\frac{3}{2}$	2	$\sqrt{\frac{1}{2}}$	0. 70711
$\frac{1}{2}$	1	$\frac{3}{2}$	2		0. 00000
$\frac{3}{2}$	1	$\frac{3}{2}$	2	$-\sqrt{\frac{1}{2}}$	-0. 70711
$\frac{5}{2}$	2	$\frac{3}{2}$	2	$\sqrt{\frac{1}{2}}$	0. 70711
$\frac{7}{2}$	2	$\frac{3}{2}$	2	$-\sqrt{\frac{1}{2}}$	-0. 70711
-1	$\frac{1}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{3}{10}}$	0. 54772
$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{3}{2}}$	0. 77460
$\frac{3}{2}$	$\frac{1}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{1}{10}}$	0. 31623
$\frac{5}{2}$	$\frac{3}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{3}{2}}$	0. 77460
$\frac{7}{2}$	$\frac{3}{2}$	1	$\frac{5}{2}$	$\sqrt{\frac{3}{2}}$	0. 63246
$\frac{5}{2}$	$\frac{5}{2}$	1	$\frac{5}{2}$		1. 00000

Compiled from A. Simon, Numerical tables of the Clebsch-Gordan coefficients, Oak Ridge National Laboratory Report 1718, Oak Ridge, Tenn. (1954) (with permission).

- [27.25] E. U. Condon and G. A. Shortley, Theory of atomic spectra (Cambridge Univ. Press, Cambridge, England, 1935).
- [27.26] M. E. Rose, Elementary theory of angular momentum (John Wiley & Sons, Inc., New York, N. Y., 1955).
- [27.27] A. Simon, Numerical tables of the Clebsch-Gordan coefficients, Oak Ridge National Laboratory Report 1718, Oak Ridge, Tenn. (1954).
 $C(j_1 j_2 j; m_1 m_2 m)$ for all angular moments $< \frac{1}{2}$, 10D.